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Constructive Approximation by Monotonous Polynomial Sequences in $\mathrm{Lip}_M \alpha$, with $\alpha \in (0, 1]$

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1. INTRODUCTION

In several papers [2–6], I have proved, for example, that every $f \in C_{[a,b]}$ can be uniformly approximated by polynomial sequences $(P_n)_n$, $(Q_n)_n$ such that

$$Q_n(x) < Q_{n+1}(x) < f(x) < P_{n+1}(x) < P_n(x),$$
$$\forall x \in [a, b], \forall n = 1, 2, ..., \tag{1}$$

(see, e.g., [3, p. 26]).

In the particular case $f \in \mathrm{Lip}_M 1$, I have obtained the sequences $(P_n)_n$ and $(Q_n)_n$ in a constructive way (see, e.g., [3, pp. 27–28]).

The purpose of this note is to construct the polynomials of (1), in the case $f \in \mathrm{Lip}_M \alpha$, $0 < \alpha \leq 1$.

The construction of $(P_n)_n$, $(Q_n)_n$ in the general case remains an open question.

2. CONSTRUCTIVE SOLUTIONS IN $\mathrm{Lip}_M \alpha$, $0 < \alpha \leq 1$

For $\alpha \in (0, 1]$, a function f: $[0, 1] \to R$ is called Hölder of class α, if there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M\, |x - y|^\alpha$, $\forall x, y \in [0, 1]$. We denote by $\mathrm{Lip}_M \alpha$ the set of all Hölder functions of the above class α and by $B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(k/n)$ the Bernstein polynomials.

THEOREM 2.1. *If $f \in \mathrm{Lip}_M \alpha$ (where $\alpha \in (0, 1]$), then for each p, a fixed integer $\geq 4/\alpha$, the polynomial sequence defined by*

$$P_n(x) = B_{n^p}(f; x) + a_n,$$

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where $a_n = 2K_0 M \sum_{j=n}^{\infty} 1/j^2$ *and* K_0 *is Sikkema's constant, satisfies* $P_n \to f$, *uniformly on* $[0, 1]$ *and* $f(x) < P_{n+1}(x) < P_n(x)$, $\forall x \in [0, 1]$, $\forall n = 1, 2, \ldots$.

Proof. It is well known [7] that if $f \in \mathrm{Lip}_M \alpha$, $\alpha \in (0, 1]$, then

$$|f(x) - B_n(f; x)| \leqslant K_0 M/(n^{\alpha/2}), \qquad \forall x \in [0, 1], \forall n = 1, 2, \ldots.$$

Hence, evidently,

$$|f(x) - B_{n^p}(f; x)| \leqslant K_0 M/(n^{p\alpha/2}), \qquad \forall x \in [0, 1], \forall n = 1, 2, \ldots.$$

Then, $|B_{n^p}(f; x) - B_{(n+1)^p}(f; x)| \leqslant |B_{n^p}(f; x) - f(x)| + |f(x) - B_{(n+1)^p}(f; x)| \leqslant K_0 M[1/(n^{p\alpha/2}) + 1/(n+1)^{p\alpha/2}] < 2K_0 M/(n^{p\alpha/2}) \leqslant 2K_0 M/n^2$, $\forall x \in [0, 1]$, $\forall n = 1, 2, \ldots$, because $p\alpha/2 \geqslant 2$ from hypothesis.

Hence, $P_{n+1}(x) - P_n(x) = B_{n^p}(f; x) - B_{(n+1)^p}(f; x) + a_n - a_{n+1} = B_{n^p}(f; x) - B_{(n+1)^p}(f; x) + 2K_0 M/n^2 > 0$, $\forall x \in [0, 1]$, $\forall n = 1, 2, \ldots$.

Also, because $a_n \searrow 0$, it follows that $P_n \to f$ uniformly on $[0, 1]$, which completes the proof.

COROLLARY 2.2. *If* $f \in \mathrm{Lip}_M \alpha$, $\alpha \in (0, 1]$, *then, for each p, a fixed integer* $\geqslant 4/\alpha$, *the polynomial sequence defined by*

$$Q_n(x) = B_{n^p}(f; x) - a_n, \qquad x \in [0, 1], n = 1, 2, \ldots,$$

satisfies $Q_n \overset{n}{\to} f$ *uniformly on* $[0, 1]$ *and* $Q_n(x) < Q_{n+1}(x) < f(x)$, $\forall x \in [0, 1]$, $\forall n = 1, 2, \ldots$.

The proof is entirely analogous.

Remarks. (1) For $\alpha = 1$, choosing $p = 4$, we obtain the construction from [3, p. 27].

(2) We have $|P_n(x) - f(x)| \leqslant |B_{n^p}(f; x) - f(x)| + a_n \leqslant K_0 M/(n^{p\alpha/2}) + a_n$, where $a_n < 1/(n-1)$, $n = 2, 3, \ldots$ (see [3, pp. 28–29]). Thus, the degree of approximation is $O(1/n^{p\alpha/2}) + O(1/(n-1))$.

For the polynomials $Q_n(x)$, we have an analogous estimate.

(3) Recently, the following result was obtained [1]: If $f \in \mathrm{Lip}_M \alpha$, then $B_n(f; x) \in \mathrm{Lip}_M \alpha$, $\forall n \geqslant 1$ ($\alpha \in (0, 1]$).

Thus, the polynomials $P_n(x)$ and $Q_n(x)$, defined in Theorem 2.1 and Corollary 2.2, have the properties $P_n(x) \in \mathrm{Lip}_M \alpha$, $Q_n(x) \in \mathrm{Lip}_M \alpha$, $\forall n = 1, 2, \ldots$.

(4) $f\colon [0, 1] \to R$ is called convex of order m ($\geqslant -1$) on $[0, 1]$ (see [8]) if, for any system of distinct points $x_i \in [0, 1]$, $i = 1, 2, \ldots, m+2$, $[x_1, \ldots, x_{m+2}; f] > 0$ (where $[x_1, \ldots, x_{m+2}; f]$ denotes the divided

difference). Also, due to T. Popoviciu [8, 9], the Bernstein polynomials preserve the convexity of order m, for any $m \geq -1$.

Because if f is convex of order m, then $f+c$ (where c is a constant) is also convex of order m for $m \geq 0$, it follows immediately that the polynomials $P_n(x), Q_n(x)$ defined in Theorem 2.1 and Corollary 2.2 preserve convexity of order $m \geq 0$.

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